On density points on the real line with respect to sequences tending to zero

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Abstract. We present the results connected with density points on the real line with respect to sequences tending to zero. The first part deals with the family od sets having the Baire property and convergence with respect to the σ-ideal of the first category sets. The second part is devoted to the family of Lebesgue measurable sets and convergence with respect to the σ-ideal of the Lebesgue null measure sets.

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1. Introduction

Through the paper we will use the standard notation: $\mathbb{R}$ will be the set of real numbers, $\mathbb{Q}$ will be the set of rational numbers and $\mathbb{N}$ the set of positive integers. By $\mathcal{B}a$ and $\mathcal{L}$ we will denote the family of Baire sets and Lebesgue measurable sets, respectively. Moreover, $\mathcal{I}$ will stand for the σ-ideal of the first category sets in $\mathbb{R}$ and $\mathbb{L}$ for the σ-ideal of the Lebesgue null measure sets. By $\lambda(A)$ we shall denote the Lebesgue measure of a measurable set A and by $|I|$ the length of an interval I. Furthermore, $\mathcal{T}_{nat}$ will denote the natural topology on $\mathbb{R}$ and $\langle s \rangle$ – an unbounded and non-decreasing sequence $\{s_n\}_{n\in\mathbb{N}}$ of positive real numbers.

We shall say that a family $\mathcal{F}$ of subsets of $\mathbb{R}$ is invariant if for every $P \in \mathcal{F}$, $x \in \mathbb{R}$ and $m \in \mathbb{R} \setminus \{0\}$ we get that $P + x \in \mathcal{F}$ and $mP \in \mathcal{F}$, where

$$P + x = \{a + x \colon a \in P\},$$
$$mP = \{ma \colon a \in P\}.$$

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According to paper [11] we shall say that 0 is a density point of a set $A \in \mathcal{B}a$ with respect to category if the sequence $\{f_n\}_{n\in\mathbb{N}} = \{\chi_{nA\cap[-1,1]}\}_{n\in\mathbb{N}}$ converges with respect to the σ-ideal $\mathcal{I}$ to the characteristic function $\chi_{[-1,1]}$. It means that every subsequence of the sequence $\{f_n\}_{n\in\mathbb{N}}$ contains a subsequence converging to the function $\chi_{[-1,1]}$ everywhere except for a set of the first category. Basing on this concept we consider more general approach.

For $J = [a,b]$ we put

$$s(J) = \frac{1}{2}(a+b),$$
$$h(A,J)(x) = \chi_{\frac{2}{|J|}(A-s(J))\cap[-1,1]}(x).$$

By $\mathcal{J}$ we shall denote a **sequence** of non-degenerate and closed intervals $\{J_n\}_{n\in\mathbb{N}}$ **tending to zero**, that means

$$\lim_{n\to\infty} s(J_n) = 0 \qquad \textit{and} \qquad \lim_{n\to\infty} |J_n| = 0.$$

These conditions are equivalent to the following one:

$$\operatorname{diam}\{J_n \cup \{0\}\} \underset{n\to\infty}{\longrightarrow} 0.$$

From now on, the family of all sequences of intervals tending to zero will be denoted by $\Im$. Moreover, we will identify sequences which differ a finite numbers of their terms. To shorten notation, we will write $\mathcal{J}$ instead of $\{J_n\}_{n\in\mathbb{N}}$.

We say that a sequence of intervals $\mathcal{J} = \{[a_n,b_n]\}_{n\in\mathbb{N}} \in \Im$, is **right-side (left-side) tending to zero** if there exists $n_0 \in \mathbb{N}$ such that $b_n > 0$ ($a_n < 0$) for $n \geqslant n_0$ and

$$\lim_{n\to\infty} \frac{\min\{0,a_n\}}{b_n} = 0 \quad \left(\lim_{n\to\infty} \frac{\max\{0,b_n\}}{a_n} = 0\right).$$

Sequence of intervals $\mathcal{J} \in \Im$ is **one-side tending to zero** if it is right-side or left-side tending to zero.

Definition 1.1. *Let $\mathcal{P}$ be a proper σ-ideal of subsets of $\mathbb{R}$ and $\mathcal{J} \in \Im$. The point 0 is a $\mathcal{P}(\mathcal{J})$-**density point** of a set $A \subset \mathbb{R}$ if*

$$h(A,J_n)(x) \underset{n\to\infty}{\overset{\mathcal{P}}{\longrightarrow}} \chi_{[-1,1]}(x),$$

it means

$$\forall\,\{n_k\}_{k\in\mathbb{N}} \quad \exists\,\{n_{k_m}\}_{m\in\mathbb{N}} \quad \exists\,\Theta \in \mathcal{P} \quad \forall\,x \notin \Theta \quad h(A,J_{n_{k_m}})(x) \underset{m\to\infty}{\longrightarrow} \chi_{[-1,1]}(x),$$

where the symbol $\underset{n\to\infty}{\overset{\mathcal{P}}{\longrightarrow}}$ stands for a convergence with respect to σ-ideal $\mathcal{P}$.

It is obvious that 0 is a $\mathcal{P}(\mathcal{J})$-density point of a set $A \subset \mathbb{R}$ if and only if

$$\forall\,\{n_k\}_{k\in\mathbb{N}} \quad \exists\,\{n_{k_m}\}_{m\in\mathbb{N}} \quad \limsup_{m\to\infty} \left([-1,1] \setminus (A - s(J_{n_{k_m}}))\frac{2}{|J_{n_{k_m}}|}\right) \in \mathcal{P}.$$

We shall say that a point $x_0 \in \mathbb{R}$ is a $\mathcal{P}(\mathcal{J})$-density point of a set $A \subset \mathbb{R}$ if and only if 0 is a $\mathcal{P}(\mathcal{J})$-density point of the set $A - x_0$.

For every set $A \subset \mathbb{R}$ let us denote

$$\Phi_{\mathcal{P}(\mathcal{J})}(A) = \{x \in \mathbb{R} : x \text{ is a } \mathcal{P}(\mathcal{J})\text{-density point of } A\}.$$

Now, let us consider invariant σ-algebra $\mathcal{S}$ of subsets of $\mathbb{R}$, $\mathcal{P}$ – a proper, invariant σ-ideal contained in $\mathcal{S}$ and $\mathcal{J} \in \Im$. As the consequence of definition of a $\mathcal{P}(\mathcal{J})$-density point we get the following theorem.

Theorem 1.2. *For every sets $A, B \in \mathcal{S}$ we have:*

1. $\Phi_{\mathcal{P}(\mathcal{J})}(\emptyset) = \emptyset, \quad \Phi_{\mathcal{P}(\mathcal{J})}(\mathbb{R}) = \mathbb{R}$;
2. $A \triangle B \in \mathcal{P} \Rightarrow \Phi_{\mathcal{P}(\mathcal{J})}(A) = \Phi_{\mathcal{P}(\mathcal{J})}(B)$;
3. $\Phi_{\mathcal{P}(\mathcal{J})}(A \cap B) = \Phi_{\mathcal{P}(\mathcal{J})}(A) \cap \Phi_{\mathcal{P}(\mathcal{J})}(B)$.

If an operator $\Phi_{\mathcal{P}(\mathcal{J})}$ satisfies an additional condition $\Phi_{\mathcal{P}(\mathcal{J})}(A) \setminus A \in \mathcal{P}$ for any $A \in \mathcal{S}$, then it is an almost lower density operator on $(\mathbb{R}, \mathcal{S}, \mathcal{P})$. Whereas, if it fulfills an additional condition $\Phi_{\mathcal{P}(\mathcal{J})}(A) \triangle A \in \mathcal{P}$ for any $A \in \mathcal{S}$, then it is a lower density operator on $(\mathbb{R}, \mathcal{S}, \mathcal{P})$.

Putting

$$\mathcal{T}_{\mathcal{P}(\mathcal{J})} = \{A \in \mathcal{S} : A \subset \Phi_{\mathcal{P}(\mathcal{J})}(A)\},$$

by conditions 1 and 2 we get that $\mathcal{T}_{\mathcal{P}(\mathcal{J})}$ is a topology if $\mathcal{S}$ coincides with the family of all subsets of real line and $\mathcal{P}$ is any proper σ-ideal. However, it does not have to be topology. For example for $\mathcal{S}$ equals the family of Borel sets on the real line, $\mathcal{P} = \{\emptyset\}$ and $\mathcal{J} = \left\{\left[\frac{1}{n}, \frac{1}{n}\right]\right\}_{n \in \mathbb{N}}$ we have that $\mathcal{T}_{\mathcal{P}(\mathcal{J})}$ is not a topology (see [6]).

In our further considerations we will concentrate only on two σ-algebras: $\mathcal{B}a$, $\mathcal{L}$ and corresponding to them σ-ideals: $\mathcal{I}$, $\mathbb{L}$. It is worth noting that, although in both cases the properties of the topologies generated by the respective density points are similar, then methods of their proving in each case are different.

2. The case of family of sets having the Baire property

In this section we will focus on the σ-ideal of the first category sets in $\mathbb{R}$, so in Definition 1.1 we will consider σ-ideal $\mathcal{I}$ instead of σ-ideal $\mathcal{P}$. In this way we will obtain an $\mathcal{I}(\mathcal{J})$-density point. Taking the special sequence $\mathcal{J} = \left\{\left[-\frac{1}{n}, \frac{1}{n}\right]\right\}_{n \in \mathbb{N}}$, we get that x_0 is an $\mathcal{I}(\mathcal{J})$-density point of a set $A \in \mathcal{B}a$ if and only if x_0 is an $\mathcal{I}$-density point of A (see [11]). Moreover, for $\langle s \rangle$ and $\mathcal{J} = \left\{\left[-\frac{1}{s_n}, \frac{1}{s_n}\right]\right\}_{n \in \mathbb{N}}$, the notion of an $\mathcal{I}(\mathcal{J})$-density point of a set $A \in \mathcal{B}a$ is equivalent to the notion of an $\langle s \rangle$-$\mathcal{I}$-density point of A (see [7]).

It should be emphasized that for an operator $\Phi_{\mathcal{I}(\mathcal{J})}$ the analogue of Lebesgue Density Theorem holds:

Theorem 2.1 ([12])**.** *For every sets $A \in \mathcal{B}a$ and $\mathcal{J} \in \Im$ we have:*

$$A \triangle \Phi_{\mathcal{I}(\mathcal{J})}(A) \in \mathcal{I}.$$

Proof. Let $A \in \mathcal{B}a$, then there exist an open set G and a set $P \in \mathcal{I}$ such that $A = G \triangle P$. We will show that $A \setminus \Phi_{\mathcal{I}(\mathcal{J})}(A) \in \mathcal{I}$. Let us take a point $x \in G$. Then there exists a number $n_0 \in \mathbb{N}$ such that $x + J_n \subset G$ for $n \geqslant n_0$, hence $J_n \subset G - x$. So we have

$$\frac{2}{|J_n|}\big(A - (x + s(J_n))\big) \supset \frac{2}{|J_n|}\left((G \setminus P) - (x + s(J_n))\right) =$$
$$= \frac{2}{|J_n|}\Big(\big((G - x) - s(J_n)\big) \setminus \big(P - (x + s(J_n))\big)\Big) \supset$$
$$\supset \frac{2}{|J_n|}\big((J_n - s(J_n)) \setminus (P - (x + s(J_n)))\big) = [-1,1] \setminus \frac{2}{|J_n|}\big(P - (x + s(J_n))\big).$$

If $P \in \mathcal{I}$ then $\frac{2}{|J_n|}\big(P - (x + s(J_n))\big) \in \mathcal{I}$. Hence for $x \in G$ we obtain that

$$h(A - x, J_n)(x) \xrightarrow[n\to\infty]{\mathcal{I}} \chi_{[-1,1]}(x),$$

so that $A \setminus \Phi_{\mathcal{I}(\mathcal{J})}(A) \subset A \setminus G \in \mathcal{I}$.

To finish the proof we must show that $\Phi_{\mathcal{I}(\mathcal{J})}(A) \setminus A \in \mathcal{I}$. Observe that $\Phi_{\mathcal{I}(\mathcal{J})}(A) \subset \mathbb{R} \setminus \Phi_{\mathcal{I}(\mathcal{J})}(\mathbb{R} \setminus A)$. Then

$$\Phi_{\mathcal{I}(\mathcal{J})}(A) \setminus A \subset (\mathbb{R} \setminus \Phi_{\mathcal{I}(\mathcal{J})}(\mathbb{R} \setminus A)) \cap (\mathbb{R} \setminus A) = (\mathbb{R} \setminus A) \setminus \Phi_{\mathcal{I}(\mathcal{J})}(\mathbb{R} \setminus A) \in \mathcal{I}.$$

□

2.1. An $\mathcal{I}(\mathcal{J})$-density topology and its properties

By Theorem 1.2 and Theorem 2.1 we have that operator $\Phi_{\mathcal{I}(\mathcal{J})}$ is a lower density operator on $(\mathbb{R}, \mathcal{B}a, \mathcal{I})$. It is well known that for any measurable space $(X, \mathcal{S}, \mathcal{P})$, where $\mathcal{S}$ is a σ-algebra of subsets of X and $\mathcal{P} \subset \mathcal{S}$ is a proper σ-ideal, if an operator $\phi : \mathcal{S} \to \mathcal{S}$ is a lower density operator on $(X, \mathcal{S}, \mathcal{P})$ and a pair $(\mathcal{S}, \mathcal{P})$ has the hull property, then the family $\mathcal{T} = \{A \in \mathcal{S} : A \subset \phi(A)\}$ is a topology (see [9]), so we have

Theorem 2.2 ([12])**.** *The family*

$$\mathcal{T}_{\mathcal{I}(\mathcal{J})} = \{A \in \mathcal{B}a : A \subset \Phi_{\mathcal{I}(\mathcal{J})}(A)\}$$

is a topology on $\mathbb{R}$*, which will be called* $\mathcal{I}(\mathcal{J})$*-density topology. Moreover,* $\mathcal{T}_{nat} \subsetneq \mathcal{T}_{\mathcal{I}(\mathcal{J})}$.

Since for any $\mathcal{J} \in \Im$, an operator $\Phi_{\mathcal{I}(\mathcal{J})}$ is a lower density operator, so by Theorem 25.3 in [9] we obtain immediately the following theorem.

Theorem 2.3. *Let* $\mathcal{J} \in \Im$.

(i) $(\mathbb{R}, \mathcal{T}_{\mathcal{I}(\mathcal{J})})$ *is a Baire space;*
(ii) $(\mathbb{R}, \mathcal{T}_{\mathcal{I}(\mathcal{J})})$ *is neither a first countable, nor a second countable, nor a separable, nor a Lindelöf space;*
(iii) $A \in \mathcal{I}$ *if and only if* A *is a closed and discrete set with respect to a topology* $\mathcal{T}$*;*
(iv) *a set* $A \subset \mathbb{R}$ *is compact with respect to a topology* $\mathcal{T}_{\mathcal{I}(\mathcal{J})}$ *if and only if* A *is finite.*

(v) $\mathcal{I}$ *is equal to the family of all meager sets with respect to a topology* $\mathcal{T}_{\mathcal{I}(\mathcal{J})}$*;*

(vi) $A \in \mathcal{B}a$ *if and only if* A *is a union of two sets - one of them is open with respect to a topology* $\mathcal{T}_{\mathcal{I}(\mathcal{J})}$ *and a second one is closed with respect to a topology* $\mathcal{T}_{\mathcal{I}(\mathcal{J})}$*;*

(vii) $\mathcal{B}a$ *coincides with the family of all Borel sets (Baire sets) with respect to a topology* $\mathcal{T}_{\mathcal{I}(\mathcal{J})}$.

Moreover, we have that

Theorem 2.4 ([13])**.** *Let* $\mathcal{J} \in \Im$*. Then* $[a, b) \in \mathcal{T}_{\mathcal{I}(\mathcal{J})}$ *(*$(a, b] \in \mathcal{T}_{\mathcal{I}(\mathcal{J})}$*) for* $a < b$ *if and only if the sequence* $\mathcal{J}$ *is right-side (left-side) tending to zero.*

This theorem yields to the following

Theorem 2.5. *If the sequence* $\mathcal{J} \in \Im$ *is right-side (left-side) tending to zero, then* $(\mathbb{R}, \mathcal{T}_{\mathcal{I}(\mathcal{J})})$ *is not connected.*

From Definition 1.1 we have the following property.

Property 2.6 ([12])**.** *Let* $\mathcal{J} \in \Im$*, then* 0 *is an* $\mathcal{I}(\mathcal{J})$*-density point of the set*

$$A_k = \{0\} \cup \bigcup_{n \geqslant k} \operatorname{int}(J_n),$$

for every $k \in \mathbb{N}$*. Moreover,* $A_k \in \mathcal{T}_{\mathcal{I}(\mathcal{J})}$.

The next theorem shows that we have obtained an essential extension of $\mathcal{I}$-density points.

Theorem 2.7 ([12])**.** *For every sequence* $\mathcal{J} \in \Im$ *there exists a sequence* $\mathcal{K} = \{K_n\}_{n\in\mathbb{N}}$ *of intervals tending to zero such that*

$$\mathcal{T}_{\mathcal{I}(\mathcal{J})} \setminus \mathcal{T}_{\mathcal{I}(\mathcal{K})} \neq \emptyset \quad \wedge \quad \mathcal{T}_{\mathcal{I}(\mathcal{K})} \setminus \mathcal{T}_{\mathcal{I}(\mathcal{J})} \neq \emptyset.$$

Proposition 2.8 ([12])**.** *Let* $\mathcal{J} = \{J_n\}_{n\in\mathbb{N}}$ *and* $\mathcal{K} = \{K_n\}_{n\in\mathbb{N}}$ *be sequences tending to zero. If for every* $n \in \mathbb{N}$ *there exists* $k(n) \in \mathbb{N}$ *such that* $J_n = K_{k(n)}$ *then* $\mathcal{T}_{\mathcal{I}(\mathcal{K})} \subset \mathcal{T}_{\mathcal{I}(\mathcal{J})}$.

The succeeding theorems gives us an examples of situation when the topologies generated by sequences of intervals are identical.

Theorem 2.9 ([12])**.** *Let* $\mathcal{J} \in \Im$ *and* $l \in \mathbb{N}$*. If we divide every interval* J_n *on equal* l *intervals and order them in a sequence* $\mathcal{K}$*, then* $\mathcal{T}_{\mathcal{I}(\mathcal{J})} = \mathcal{T}_{\mathcal{I}(\mathcal{K})}$.

Theorem 2.10 ([12])**.** *Let* $\mathcal{J} = \{J_n\}_{n\in\mathbb{N}}$ *and* $\mathcal{K} = \{K_n\}_{n\in\mathbb{N}}$ *be sequences of intervals tending to zero. If*

$$\lim_{n\to\infty} \frac{\lambda(J_n \vartriangle K_n)}{\lambda(J_n \cap K_n)} = 0,$$

then $\mathcal{T}_{\mathcal{I}(\mathcal{J})} = \mathcal{T}_{\mathcal{I}(\mathcal{K})}$.

The following theorems show some properties of the family of $\mathcal{I}(\mathcal{J})$ type topologies.

Theorem 2.11 ([12])**.** *Let* $\mathcal{T}_H = \{V \setminus P : V \in \mathcal{T}_{nat} \wedge P \in \mathcal{I}\}$*. Then*

$$\bigcap_{\mathcal{J} \in \Im} \mathcal{T}_{\mathcal{I}(\mathcal{J})} = \mathcal{T}_H.$$

Theorem 2.12 ([12]). *Let $\mathcal{T}^*$ be the topology generated by $\bigcup_{\mathcal{J}\in\Im}\mathcal{T}_{\mathcal{I}(\mathcal{J})}$. Then*

$$\mathcal{T}^* = 2^{\mathbb{R}} \quad \text{and} \quad \bigcup_{\mathcal{J}\in\Im}\mathcal{T}_{\mathcal{I}(\mathcal{J})} \neq \mathcal{T}^*.$$

Theorem 2.13 ([12]). *For any sequence $\mathcal{J}\in\Im$, the space $(\mathbb{R},\mathcal{T}_{\mathcal{I}(\mathcal{J})})$ is Hausdorff but not regular.*

2.2. $\mathcal{I}(\mathcal{J})$-approximately continuous functions

The class of approximately continuous function was defined by Denjoy in [3]. The category analogue of approximate continuity was presented by Poreda, Wagner-Bojakowska and Wilczyński in [11].

A function $f\colon\mathbb{R}\to\mathbb{R}$ is $\mathcal{I}(\mathcal{J})$-**approximately continuous**, if it is continuous with respect to the $\mathcal{I}(\mathcal{J})$-density topology on the domain and the natural topology on the range.

Theorem 2.14 ([13]). *Let $\mathcal{J}$ be a sequence of intervals tending to zero. Then every $\mathcal{I}(\mathcal{J})$-approximately continuous function is of the first Baire class.*

In the proof of the last theorem of this subsection, the following easy fact is needed.

Conclusion 2.15 ([13]). *If a sequence $\mathcal{J}\in\Im$ is not one-side tending to zero, then for any nonempty set $U\in\mathcal{T}_{\mathcal{I}(\mathcal{J})}$, $\delta>0$ and $x\in\Phi_{\mathcal{I}(\mathcal{J})}(U)$ we have that*

$$U\cap(x,x+\delta)\neq\emptyset \quad \text{and} \quad U\cap(x-\delta,x)\neq\emptyset.$$

Theorem 2.16 ([13]). *Let $\mathcal{J}\in\Im$. Every $\mathcal{I}(\mathcal{J})$-approximately continuous function is Darboux function if and only if the sequence $\mathcal{J}$ is not one-side tending to zero.*

Proof. Necessity. Let the sequence $\mathcal{J}\in\Im$ is right-side tending to zero and define function

$$f(x) = x-k \qquad \text{for } x\in[k,k+1).$$

This is an $\mathcal{I}(\mathcal{J})$-approximately continuous function but it is not a Darboux function.

If the sequence $\mathcal{J}\in\Im$ is left-side tending to zero, then we consider function

$$g(x) = x-k \qquad \text{for } x\in(k,k+1].$$

Sufficiency. Let the sequence $\mathcal{J}\in\Im$ be not one-side tending to zero and f be an $\mathcal{I}(\mathcal{J})$-approximately continuous function. Fix an $x\in\mathbb{R}$ and for each $n\in\mathbb{N}$ define the set

$$V_n = \left(f(x)-\frac{1}{n}, f(x)+\frac{1}{n}\right).$$

By $\mathcal{I}(\mathcal{J})$-approximate continuity of the function f there exists a set $U_n\in\mathcal{T}_{\mathcal{I}(\mathcal{J})}$ such that $f(U_n)\subset V_n$ and $x\in U_n$ is an $\mathcal{I}(\mathcal{J})$-density point of the set U_n. Conclusion 2.15 implies that for any $n\in\mathbb{N}$ there exist

$$y_n^1 \in U_n \cap \left(x - \frac{1}{n}, x\right) \quad \text{and} \quad y_n^2 \in U_n \cap \left(x, x + \frac{1}{n}\right).$$

Hence we obtain two sequences $\left\{y_n^1\right\}_{n\in\mathbb{N}}$ and $\left\{y_n^2\right\}_{n\in\mathbb{N}}$ such that

$$y_n^1 \underset{n\to\infty}{\longrightarrow} x \quad \text{and} \quad y_n^2 \underset{n\to\infty}{\longrightarrow} x$$

and

$$\lim_{n\to\infty} f(y_n^1) = \lim_{n\to\infty} f(y_n^2).$$

Obviously, by Theorem 2.14 we have that f is of the first Baire class. Thus, by Young's criterion (Theorem 1.1 in [1]), we conclude that function f is Darboux function. □

3. The case of the family of Lebesgue measurable sets

The notion of a density point connected with the Lebesgue measure was introduced at the beginning of 20th century. We say that $x_0 \in \mathbb{R}$ is a density point of a Lebesgue measurable set A if

$$\lim_{h\to 0^+} \frac{\lambda(A \cap [x_0 - h, x_0 + h])}{2h} = 1. \tag{1}$$

Moreover, we have that the condition (1) can be replaced with the following one:

$$\lim_{n\to\infty} \frac{\lambda(A \cap [x_0 - \frac{1}{n}, x_0 + \frac{1}{n}])}{\frac{2}{n}} = 1. \tag{2}$$

This observation led to the generalization of a density point introduced in 2004 by M. Filipczak and J. Hejduk ([5]). They considered in (2) a sequence $\langle s\rangle$ instead of the sequence $\{n\}_{n\in\mathbb{N}}$. Thus we say that $x_0 \in \mathbb{R}$ is an $\langle s\rangle$-density point of a set $A \in \mathcal{L}$ if

$$\lim_{n\to\infty} \frac{\lambda(A \cap [x_0 - \frac{1}{s_n}, x_0 + \frac{1}{s_n}])}{\frac{2}{s_n}} = 1. \tag{3}$$

Moreover we have that the condition (2) is equivalent the condition (3) iff $\liminf_{n\to\infty} \frac{s_n}{s_{n+1}} > 0$ (see [5]).

One can ask what will happen if we replace the sequence $\{[x_0 - \frac{1}{s_n}, x_0 + \frac{1}{s_n}]\}_{n\in\mathbb{N}}$ by a sequence $\mathcal{J} + x_0$, where $\mathcal{J} \in \Im$? In this case we also obtain a new kind of a density poin – a $\mathcal{J}$-density point. This notion was introduced in 2013 by R. Wiertelak.

Definition 3.1. *Let $A \in \mathcal{L}$, $\mathcal{J} \in \Im$. We shall say that the point* 0 *is a $\mathcal{J}$-**density point** of a set A if*

$$\lim_{n\to\infty} \frac{\lambda(A \cap J_n)}{|J_n|} = 1. \tag{4}$$

Note that in the general case of a $\mathcal{P}(J)$-density point we consider a special sequence of characteristic functions converges to $\chi_{[-1,1]}$ with respect to σ-ideal $\mathcal{P}$ (see Definition 1.1). It is worth adding that in the case of a $\mathcal{J}$-density point we can also consider such sequence. More specifically, an equivalent formulation of (4) is:

$$\chi_{\frac{2}{|J_n|}(A-s(J_n))\cap[-1,1]}(x)\xrightarrow[n\to\infty]{\lambda}\chi_{[-1,1]}(x),$$

where the symbol $\xrightarrow[n\to\infty]{\lambda}$ denotes a convergence with respect to the Lebesgue measure. Indeed, the following equivalences are obvious.

$$\lim_{n\to\infty}\frac{\lambda(A\cap J_n)}{|J_n|}=1\Leftrightarrow\lim_{n\to\infty}\frac{\lambda((A-s(J_n))\cap(J_n-s(J_n)))}{|J_n|}=1\Leftrightarrow$$
$$\lim_{n\to\infty}\frac{2}{|J_n|}\lambda((A-s(J_n))\cap(J_n-s(J_n)))=2\Leftrightarrow$$
$$\lim_{n\to\infty}\lambda(\frac{2}{|J_n|}(A-s(J_n))\cap\frac{2}{|J_n|}(J_n-s(J_n)))=2\Leftrightarrow$$
$$\lim_{n\to\infty}\lambda(\frac{2}{|J_n|}(A-s(J_n))\cap[-1,1])=2\Leftrightarrow\chi_{\frac{2}{|J_n|}(A-s(J_n))\cap[-1,1]}(x)\xrightarrow[n\to\infty]{\lambda}\chi_{[-1,1]}(x).$$

In addition, it is worth noting that the last condition saved in the last line above is equivalent to the following

$$\forall\{n_k\}_{k\in\mathbb{N}}\quad\exists\{n_{k_j}\}_{j\in\mathbb{N}}\quad\chi_{\frac{2}{|J_{n_{k_j}}|}(A-s(J_{n_{k_j}}))\cap[-1,1]}(x)\xrightarrow[j\to\infty]{}\chi_{[-1,1]}(x)\quad\mathbb{L}\text{ a.e.}$$

where $\mathbb{L}$ a.e. means that in this case we consider $\mathbb{L}$-almost everywhere convergence.

Thus in the case of the Lebesgue measure we can check whether a point 0 is a $\mathcal{J}$-density point of a set $A\in\mathcal{L}$, as in the case of $\mathcal{I}(J)$-density point. However, it appears that in the case of the Lebesgue measure the condition (4) is easier to check and it is more often applied.

Obviously, a point $x_0\in\mathbb{R}$ is a $\mathcal{J}$-density point of a set $A\in\mathcal{L}$ if 0 is a $\mathcal{J}$-density point of a set $A-x_0$ or equivalently if

$$\lim_{n\to\infty}\frac{\lambda(A\cap(x_0+J_n))}{|J_n|}=1.$$

If for any $A\in\mathcal{L}$ and $\mathcal{J}\in\Im$ we put

$$\Phi_{\mathcal{J}}(A)=\{x\in\mathbb{R}: x\text{ is a }\mathcal{J}\text{-density point of the set }A\},$$

then we obtain that $\Phi_{\mathcal{J}}(A)\in\mathcal{L}$ for any $A\in\mathcal{L}$ (see [10]) and operator $\Phi_{\mathcal{J}}:\mathcal{L}\to\mathcal{L}$ has properties presented in Theorem 1.2 for $\mathcal{P}=\mathbb{L}$. It is also worth noting that theorem analogous to Theorem 2.1 is not true for every sequence $\mathcal{J}\in\Im$. In [2] there is a construction of a set $A\in\mathcal{L}$ and a sequence $\mathcal{J}\in\Im$ such that $\Phi_{\mathcal{J}}(A)\bigtriangleup A\notin\mathbb{L}$. However, if we consider a subfamily $\Im_\alpha\subset\Im$ such that for any sequence $\mathcal{J}\in\Im_\alpha$ we have

$$\alpha(\mathcal{J})=\limsup_{n\to\infty}\frac{\operatorname{diam}(J_n\cup\{0\})}{|J_n|}<\infty,$$

then the analogue of Lebesgue Density Theorem holds.

Theorem 3.2 ([10])**.** *If $\mathcal{J}\in\Im_\alpha$ and $A\in\mathcal{L}$ then $\Phi_{\mathcal{J}}(A)\bigtriangleup A\in\mathbb{L}$.*

Proof. We only need to show that $A \setminus \Phi_{\mathcal{J}}(A) \in \mathbb{L}$ for any bounded set A. Moreover, there is no loss of generality in assuming that the sequence $\{|J_n|\}_{n\in\mathbb{N}}$ is decreasing and

$$\operatorname{diam}\{\{0\} \cup J_n\} < 2\alpha(\mathcal{J})|J_n|. \tag{5}$$

First, we will prove that for any $0 < \varepsilon < 1$

$$E_\varepsilon = \left\{x \in A\colon \liminf_{n\in\mathbb{N}} \frac{\lambda(A \cap (J_n + x))}{|J_n|} < 1 - \varepsilon\right\} \in \mathbb{L}. \tag{6}$$

Suppose, contrary to our claim, that the outer Lebesgue measure of E, denoted by $\lambda^*(E)$, is greater than 0. Thus one can find a set $G \in \mathcal{T}_{nat}$ such that $E_\varepsilon \subset G$ and $(1-\varepsilon)\lambda(G) < \lambda^*(E_\varepsilon)$.

Let $\mathcal{E}$ be the family of all closed intervals $I \subset G$ such that $\lambda(A \cap I) < (1-\varepsilon)|I|$ and $I = J_n + x$ for some $x \in E_\varepsilon$ and $n \in \mathbb{N}$. Observe that

(i) every neighbourhood of each $x \in E_\varepsilon$ contains an interval $I \in \mathcal{E}$;
(ii) for any sequence $\{I_n\}$ of disjoint intervals from $\mathcal{E}$ the inequality $\lambda^*\left(E_\varepsilon \setminus \bigcup I_n\right) > 0$ holds.

The property (i) is obvious. The property (ii) results from the following fact

$$\lambda^*\Big(E_\varepsilon \cap \bigcup_{n\in\mathbb{N}} I_n\Big) \leqslant \sum_{n\in\mathbb{N}} \lambda(A \cap I_n) \leqslant (1-\varepsilon)\sum_{n\in\mathbb{N}} |I_n| =$$
$$= (1-\varepsilon)\lambda\Big(\bigcup_{n\in\mathbb{N}} I_n\Big) \leqslant (1-\varepsilon)\lambda(G) < (1-\varepsilon)\lambda^*(E_\varepsilon).$$

Now, we will construct inductively a sequence $\{I_n\}_{n\in\mathbb{N}}$ of disjoint intervals from $\mathcal{E}$. We start by putting

$$k_0 = \min\left\{i \in \mathbb{N}\colon \underset{x\in E_\varepsilon}{\exists} \quad J_i + x \in \mathcal{E}\right\}$$

and choosing interval I_1 from $\mathcal{E}$ such that $|I_1| = |J_{k_0}|$. Assume that intervals I_i for $i \in \{1, 2, \ldots, n\}$ have been chosen. Let $\mathcal{E}_n$ be the subset of $\mathcal{E}$ which consists of all intervals that are disjoint from $I_1, \ldots, I_n$. Properties (ii) and (i) imply that $\mathcal{E}_n \neq \emptyset$. Define

$$k_n = \min\left\{i \in \mathbb{N}\colon \underset{x\in E_\varepsilon}{\exists} \quad J_i + x \in \mathcal{E}_n\right\}.$$

and choose an interval I_{n+1} from $\mathcal{E}_n$ with length $|J_{k_n}|$.

Putting $B = E_\varepsilon \setminus \bigcup\limits_{n\in\mathbb{N}} I_n$ we obtain, by (ii), that $\lambda^*(B) > 0$. Hence there is $N \in \mathbb{N}$ such that

$$\sum_{n=N+1}^{\infty} |I_n| < \frac{\lambda^*(B)}{4\alpha(\mathcal{J}) + 1}. \tag{7}$$

For each $n > N$ let K_n be the interval concentric with I_n such that $|K_n| = (4\alpha(\mathcal{J}) + 1)|I_n|$. The inequality (7) implies that $\bigcup\limits_{n\in\mathbb{N}} K_n$ does not cover the set B, so there exists a point

$$x \in B \setminus \bigcup_{n>N} K_n. \tag{8}$$

Therefore $x \in E_\varepsilon \setminus \bigcup_{n=1}^{N} I_n$. From (ii) and (i) it follows that there exists an interval $I_x \in \mathcal{E}_N$ such that $I_x = J_{n_x} + x$ for some $n_x \in \mathbb{N}$. It is clear that $I_x \cap I_n \neq \emptyset$ for some $n > N$. Putting $n_0 = \min\{n \in \mathbb{N}\colon I_n \cap I_x \neq \emptyset\}$ we obtain that $|J_{n_x}| = |I_x| \leqslant |I_{n_0}|$. The condition (5) implies

$$\operatorname{dist}(x, I_{n_0}) \leqslant \operatorname{diam}\{x \cup I_x\} = \operatorname{diam}\{\{0\} \cup J_{n_x}\} < 2\alpha(\mathcal{J})|J_{n_x}| \leqslant 2\alpha(\mathcal{J})|I_{n_0}|,$$

where $\operatorname{dist}(x, I_{n_0})$ denotes the distance between the point x and the interval I_{n_0}. Thus $x \in K_{n_0}$, contrary to (8).

From (6) and the inclusion

$$A \setminus \Phi_{\mathcal{J}}(A) \subset \bigcup_{\varepsilon \in (0,1)\cap\mathbb{Q}} E_\varepsilon$$

we get immediately that $A \vartriangle \Phi_{\mathcal{J}}(A) \in \mathbb{L}$. □

Therefore for any $\mathcal{J} \in \Im_\alpha$ an operator $\Phi_{\mathcal{J}}$ is a lower density operator on $(\mathbb{R}, \mathcal{L}, \mathbb{L})$. In addition, it is worth noting that when we compare the above proof with proof of Theorem 2.1, it is easy to observe differences in the methods that are used in them. We see at once that in the case of $\mathcal{J}$-density points we need to take other action than in the case of $\mathcal{I}(\mathcal{J})$-density points.

Obviously, one can ask what will happen if we consider any $\mathcal{J} \in \Im$. In this case we can prove the following fact.

Theorem 3.3 ([10]). *If $\mathcal{J} \in \Im$ and $A \in \mathcal{L}$ then $\Phi_{\mathcal{J}}(A) \setminus A \in \mathbb{L}$.*

Therefore for any $\mathcal{J} \in \Im$ an operator $\Phi_{\mathcal{J}}$ is an almost lower density operator on $(\mathbb{R}, \mathcal{L}, \mathbb{L})$.

3.1. A $\mathcal{J}$-density topology and its property

In this section we will focus our attention on topology generated by $\mathcal{J}$-density points. As in the case of $\mathcal{I}(\mathcal{J})$-density topology (see Section 2.1) we have that family $\mathcal{T}_{\mathcal{J}} = \{A \in \mathcal{L} : A \subset \Phi_{\mathcal{J}}(A)\}$ is a topology for any $\mathcal{J} \in \Im_\alpha$.

What is more, in [9] one can find the property that for any measurable space $(X, \mathcal{S}, \mathcal{P})$, where $\mathcal{S}$ is a σ-algebra of subsets of X and $\mathcal{P} \subset \mathcal{S}$ is a proper σ-ideal, if an operator $\phi : \mathcal{S} \to \mathcal{S}$ is an almost lower density operator on $(X, \mathcal{S}, \mathcal{P})$ and a pair $(\mathcal{S}, \mathcal{P})$ has the hull property, then the family $\mathcal{T} = \{A \in \mathcal{S} : A \subset \phi(A)\}$ is a topology, so we immediately obtain

Theorem 3.4 ([10]). *Let $\mathcal{J} \in \Im$. The family*

$$\mathcal{T}_{\mathcal{J}} = \{A \in \mathcal{L} : A \subset \Phi_{\mathcal{J}}(A)\}$$

is a topology called a $\mathcal{J}$-density topology.

Moreover, as in the case of $\mathcal{I}(\mathcal{J})$-density topology, it is easy to see that $\mathcal{T}_{nat} \subsetneq \mathcal{T}_{\mathcal{J}}$. Furthermore, since for any $\mathcal{J} \in \Im$, an operator $\Phi_{\mathcal{J}}$ is an almost lower density operator, so by Theorem 25.27 in [9] we obtain immediately the following claim

Theorem 3.5. *Let $\mathcal{J} \in \Im$.*

(i) $(\mathbb{R}, \mathcal{T}_{\mathcal{J}})$ is neither a first countable, nor a second countable, nor a separable, nor a Lindelöf space;
(ii) $A \in \mathbb{L}$ if and only if A is a closed and discrete set with respect to a topology $\mathcal{T}_{\mathcal{J}}$;
(iii) a set $A \subset \mathbb{R}$ is compact with respect to a topology $\mathcal{T}_{\mathcal{J}}$ if and only if A is finite.

If $\mathcal{J} \in \Im_\alpha$, then an operator $\Phi_{\mathcal{J}}$ is a lower density operator, so in this case to the properties presented in Theorem 3.5 we can add another (cf. Theorem 25.3 in [9])

Theorem 3.6. *Let $\mathcal{J} \in \Im_\alpha$.*

(a) $(\mathbb{R}, \mathcal{T}_{\mathcal{J}})$ is a Baire space;
(b) $\mathbb{L}$ is equal to the family of all meager sets with respect to a topology $\mathcal{T}_{\mathcal{J}}$;
(c) $A \in \mathcal{L}$ if and only if A is a union of two sets - one of them is open with respect to a topology $\mathcal{T}_{\mathcal{J}}$ and a second one is closed with respect to a topology $\mathcal{T}_{\mathcal{J}}$;
(d) $\mathcal{L}$ coincides with the family of all Borel sets (Baire sets) with respect to a topology $\mathcal{T}_{\mathcal{J}}$.

One can ask about the connection between the density topology $\mathcal{T}_d$ and a $\mathcal{J}$-density topology. If we consider an unbounded and nondecreasing sequence $\{s_n\}_{n\in\mathbb{N}}$ of positive numbers and a sequence $\mathcal{J} = \{J_n\}_{n\in\mathbb{N}}$, where $J_n = [-\frac{1}{s_n}, \frac{1}{s_n}]$ for $n \in \mathbb{N}$, then we have that $\mathcal{T}_d \subset \mathcal{T}_{\mathcal{J}}$ (see [4]). In general, such a relationship does not have to take place. Indeed, we have

Theorem 3.7 ([10]). *If $\mathcal{J} \in \Im \setminus \Im_\alpha$, then there exists an open set A such that $0 \in \Phi_d(A)$ and $0 \notin \Phi_{\mathcal{J}}(A)$.*

From the above theorem we can deduce at once

Theorem 3.8. *If $\mathcal{J} \in \Im \setminus \Im_\alpha$, then $\mathcal{T}_d \setminus \mathcal{T}_{\mathcal{J}} \neq \emptyset$.*

Moreover, we can show that there exists a sequence $\mathcal{J} \in \Im \setminus \Im_\alpha$ such that $\mathcal{T}_{\mathcal{J}} \setminus \mathcal{T}_d \neq \emptyset$ and $\mathcal{T}_d \setminus \mathcal{T}_{\mathcal{J}} \neq \emptyset$. However, if we consider any sequence $\mathcal{J} \in \Im_\alpha$, then $\mathcal{T}_d \setminus \mathcal{T}_{\mathcal{J}} = \emptyset$. We can describe the relationship between the density topology and $\mathcal{T}_{\mathcal{J}}$ in the following way.

Theorem 3.9 ([10]). *Let $\mathcal{J} \in \Im$. The following conditions are equivalent:*

(a) $\alpha(\mathcal{J}) < +\infty$;
(b) $\mathcal{T}_d \subset \mathcal{T}_{\mathcal{J}}$.

It should be added that there are sequences $\mathcal{J},\mathcal{K} \in \Im_\alpha$ such that $\mathcal{T}_d \neq \mathcal{T}_\mathcal{J}$ and $\mathcal{T}_d = \mathcal{T}_\mathcal{K}$.

Interesting is also the question about the relationship between the $\mathcal{J}$-density topologies for different sequences $\mathcal{J} \in \Im$. The question whether the claim analogous to Theorem 2.7 is true in the case of the $\mathcal{J}$- density topology is still open. However, we have

Theorem 3.10. *There exist sequences $\mathcal{K},\mathcal{J} \in \Im$ such that $\mathcal{T}_\mathcal{J} \setminus \mathcal{T}_\mathcal{K} \neq \emptyset$, $\mathcal{T}_\mathcal{K} \setminus \mathcal{T}_\mathcal{J} \neq \emptyset$, $\mathcal{T}_\mathcal{J} \neq \mathcal{T}_d$ and $\mathcal{T}_\mathcal{K} \neq \mathcal{T}_d$.*

To prove this it suffices to consider sequences $\mathcal{J} = \{[-\frac{1}{(2n-1)!}, \frac{1}{(2n-1)!}]\}_{n\in\mathbb{N}}$ and $\mathcal{K} = \{[-\frac{1}{(2n)!}, \frac{1}{(2n)!}]\}_{n\in\mathbb{N}}$ (see [4]).

The next theorem shows some connection between $\mathcal{J}$-density topology and $\langle s\rangle$-density topology associated with $\langle s\rangle$-density points.

Theorem 3.11 ([10])**.** *If $\mathcal{J} \in \Im_\alpha$, then there exists a sequence $\mathcal{K} \in \Im$ of symmetrical intervals such that $\mathcal{T}_\mathcal{K} \subset \mathcal{T}_\mathcal{J}$.*

Moreover, from Theorem 3.8 and 3.9 it may be concluded

Theorem 3.12. *If $\mathcal{J} \in \Im_\alpha$ and $\mathcal{K} \in \Im \setminus \Im_\alpha$, then $\mathcal{T}_\mathcal{J} \setminus \mathcal{T}_\mathcal{K} \neq \emptyset$.*

Furthermore, Theorem 8 in [5] gives

Theorem 3.13. *Let $\mathcal{T}^*$ be the topology generated by $\bigcup_{\mathcal{J}\in\Im} \mathcal{T}_\mathcal{J}$. Then*

$$\mathcal{T}^* = 2^\mathbb{R} \quad \text{and} \quad \bigcup_{\mathcal{J}\in\Im} \mathcal{T}_\mathcal{J} \neq \mathcal{T}^*.$$

We end this section with two properties connected with separation axioms for $\mathcal{J}$-density topology. The second one will show the differences between the $\mathcal{J}$-density topology and $\mathcal{I}(\mathcal{J})$-density topology for $\mathcal{J} \in \Im_\alpha$.

Since for any $\mathcal{J} \in \Im$, we have that $\mathcal{T}_{nat} \subset \mathcal{T}_\mathcal{J}$, so

Property 3.14 ([8])**.** *For any $\mathcal{J} \in \Im$ a space $(\mathbb{R}, \mathcal{T}_\mathcal{J})$ is a Hausdorff space.*

Moreover, we have

Property 3.15 ([8])**.** *For any $\mathcal{J} \in \Im_\alpha$ a space $(\mathbb{R}, \mathcal{T}_\mathcal{J})$ is regular and it is not normal.*

The question whether for any sequence $\mathcal{J} \in \Im_\alpha$ a space $(\mathbb{R}, \mathcal{T}_\mathcal{J})$ is completely regular is still open. Just like the question whether for any $\mathcal{J} \in \Im \setminus \Im_\alpha$ a space $(\mathbb{R}, \mathcal{T}_\mathcal{J})$ is regular, completely regular or normal.

3.2. $\mathcal{J}$-approximately continuous functions

It was mentioned in Section 2.2 that the notion of approximately continuous functions was introduced by Denjoy. He considered this class of functions in conjunction with the density points. Now, we will concentrate on the analogue concept in connection with $\mathcal{J}$-density points.

Let $f : \mathbb{R} \to \mathbb{R}$ and $\mathcal{J} \in \Im$. We say that f is **$\mathcal{J}$-approximately continuous at a point** $x_0 \in \mathbb{R}$ if there exists a set $A_{x_0} \in \mathcal{L}$ such that $x_0 \in \Phi_{\mathcal{J}}(A_{x_0})$ and $f(x_0) = \lim_{\substack{x \to x_0, \\ x \in A_{x_0}}} f(x)$.

Obviously, we say that $f : \mathbb{R} \to \mathbb{R}$ is a **$\mathcal{J}$-approximately continuous function** if it is $\mathcal{J}$-approximately continuous at each point $x \in \mathbb{R}$. It is easy to see that, if $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, then it is $\mathcal{J}$-approximately continuous for any sequence $\mathcal{J} \in \Im$. If $f : \mathbb{R} \to \mathbb{R}$ is an approximately continuous function, then it is $\mathcal{J}$-approximately continuous for any sequence $\mathcal{J} \in \Im_\alpha$. For any $\mathcal{J} \in \Im \setminus \Im_\alpha$ there exists an approximately continuous function $f : \mathbb{R} \to \mathbb{R}$ which is not $\mathcal{J}$-approximately continuous (see [8]).

Theorem 3.16 ([8])**.** *Let $\mathcal{J} \in \Im$. The family of all $\mathcal{J}$-approximately continuous functions $f : \mathbb{R} \to \mathbb{R}$ is closed under addition and multiplication. Moreover, if $g : \mathbb{R} \to \mathbb{R}$ is $\mathcal{J}$-approximately continuous, then the function $\frac{1}{g}$ is $\mathcal{J}$-approximately continuous, whenever $g(x) \neq 0$ for any $x \in \mathbb{R}$.*

Now, we will focus our attention on sequences $\mathcal{J}$ belonging to $\Im_\alpha$. The question whether the following statements are true also for sequences $\mathcal{J} \in \Im \setminus \Im_\alpha$ is still open.

Theorem 3.17 ([8])**.** *Let $\mathcal{J} \in \Im_\alpha$. If $f : \mathbb{R} \to \mathbb{R}$ is a $\mathcal{J}$-approximately continuous function, then f is of the first Baire class.*

The relationship between the $\mathcal{J}$-approximately continuous functions and $\mathcal{J}$-density topology for $\mathcal{J} \in \Im_\alpha$ can be explained in the following theorem.

Theorem 3.18 ([8])**.** *Let $f : \mathbb{R} \to \mathbb{R}$ and $\mathcal{J} \in \Im_\alpha$. The function f is $\mathcal{J}$-approximately continuous if and only if for any $\beta \in \mathbb{R}$ the sets $\{x \in \mathbb{R} : f(x) < \beta\}$ and $\{x \in \mathbb{R} : f(x) > \beta\}$ belong to the topology $\mathcal{T}_{\mathcal{J}}$.*

In addition, there is a relationship between these functions and Lebesgue measurable functions analogous to the case of approximately continuous functions.

Theorem 3.19 ([8])**.** *Let $f : \mathbb{R} \to \mathbb{R}$. The following conditions are equivalent:*

(i) f is a Lebesgue measurable function,
(ii) there exists $B \in \mathbb{L}$ such that for any sequence $\mathcal{J} \in \Im_\alpha$ and any $x \in \mathbb{R} \setminus B$ the function f is $\mathcal{J}$-approximately continuous at a point x,
(iii) there exists a sequence $\mathcal{J} \in \Im_\alpha$ and there exists $B_{\mathcal{J}} \in \mathbb{L}$ such that the function f is $\mathcal{J}$-approximately continuous at each point $x \in \mathbb{R} \setminus B_{\mathcal{J}}$.

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